

# Mean value property associated with the Dunkl Laplacian

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## Abstract

Let  $\Delta_k$  be the Dunkl Laplacian on  $\mathbb{R}^d$ . The main goal of this paper is to characterize  $\Delta_k$ -harmonic functions by means of a mean value property.

**Keywords :** Dunkl Laplacian, Mean value property,  $\Delta_k$ -harmonic functions.

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## 1 Introduction

Let  $R$  be a root system of  $\mathbb{R}^d$ ,  $d \geq 1$ ,  $k : R \rightarrow \mathbb{R}_+$  be a multiplicity function and  $W$  be the group generated by the reflections  $\sigma_\alpha$ ,  $\alpha \in R$ . The Dunkl Laplacian is defined in [1] for every function  $f \in C^2(\mathbb{R}^d)$  by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left( \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{|\alpha|^2}{2} \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),$$

where  $\Delta$  and  $\nabla$  denote respectively the usual Laplacian and gradient on  $\mathbb{R}^d$  and  $R_+$  is a positive subsystem of  $R$ . Clearly, if  $k$  is the identically vanishing function, then  $\Delta_k$  is reduced to  $\Delta$ .

It is well known that a locally bounded function  $f$  on an open subset  $D$  of  $\mathbb{R}^d$ , is  $\Delta$ -harmonic (i.e.,  $f \in C^2(D)$  and  $\Delta f = 0$  on  $D$ ) if and only if

$$f(x) = \frac{1}{\sigma_{x,r}(S(x,r))} \int_{S(x,r)} f(y) d\sigma_{x,r}(y),$$

for every  $x \in D$  and every  $r > 0$  such that the closed ball  $\overline{B}(x, r)$  of center  $x$  and radius  $r$  is contained in  $D$ . Here  $\sigma_{x,r}$  is the surface area measure on the sphere  $S(x, r)$  with center  $x$  and radius  $r$ .

H. Mejjiaolli and K. Trimèche showed in [4] that every infinitely differentiable function  $f$  on  $\mathbb{R}^d$  is  $\Delta_k$ -harmonic on  $\mathbb{R}^d$  if and only if for all  $x \in \mathbb{R}^d$  and  $r > 0$ ,

$$f(x) = \frac{1}{d_k} \int_{S(0,1)} \tau_x f(ry) \left( \prod_{\alpha \in R_+} |\langle y, \alpha \rangle|^{2k(\alpha)} \right) d\sigma_{0,1}(y), \quad (1)$$

where  $d_k$  is a normalized constant and  $\tau_x$  is the Dunkl translation. The main goal of this paper is to investigate a mean value property which characterizes the  $\Delta_k$ -harmonicity of locally bounded functions on an open subset of  $\mathbb{R}^d$ .

Let  $D \subset \mathbb{R}^d$  be an open set which is  $W$ -invariant. We shall say that a function  $f : D \rightarrow \mathbb{R}$  satisfies the mean value property on  $D$  if for every  $x \in D$  and  $r > 0$  such that  $\overline{B}(x, r) \subset D$ ,

$$f(x) = \int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y),$$

where  $\sigma_{x,r}^k$  (see [7]) is the unique probability measure on  $\mathbb{R}^d$  such that the right hand side of (1) coincides with

$$\int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y).$$

We shall prove that every locally bounded function  $f$  on  $D$  is  $\Delta_k$ -harmonic if and only if it satisfies the mean value property on  $D$ . To that end, we prove first the equivalence for infinitely differentiable functions on  $D$ . Next, we show that for a locally bounded function  $f$  on  $D$ , if  $f$  satisfies the mean value property then  $f$  is infinitely differentiable on  $D$ . Thus,  $f$  is  $\Delta_k$ -harmonic provided it satisfies the mean value property on  $D$ . To prove the converse, we need only show that if  $f$  is  $\Delta_k$ -harmonic then it is infinitely differentiable on  $D$ . This will be proved once we have shown that the operator  $\Delta_k$  is hypoelliptic. Thus, by means of convergence property of  $\Delta_k$ -harmonic functions, we prove that the operator  $\Delta_k$  is hypoelliptic on  $D$ .

Note that the condition that  $D$  is  $W$ -invariant is nearly optimal. In fact, in the case where  $d = 1$ , for every open set  $D \subset \mathbb{R}$  which is not  $W$ -invariant, we can always construct a  $\Delta_k$ -harmonic function  $f$  on  $D$  which does not satisfy the mean value property on  $D$ .

## 2 Preliminaries and some lemmas

Let  $S(\mathbb{R}^d)$  be the Schwartz space and  $C_0(\mathbb{R}^d)$  be the set of all continuous functions on  $\mathbb{R}^d$  vanishing at infinity. For every open set  $U \subset \mathbb{R}^d$ ,  $C(U)$  and  $C_c(U)$  will denote respectively the set of all continuous functions on  $U$  and the set of all continuous functions with compact support on  $U$ . The

set of all bounded functions in  $C(U)$  will be denoted by  $C_b(U)$ . For every  $\alpha \in \mathbb{R}^d \setminus \{0\}$ , let  $H_\alpha$  be the hyperplane of  $\mathbb{R}^d$  orthogonal to  $\alpha$  and let  $\sigma_\alpha$  be the reflection in  $H_\alpha$ , i.e.,

$$\sigma_\alpha(x) := x - 2 \frac{\langle \alpha, x \rangle}{|\alpha|^2} \alpha,$$

where  $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$  and  $|x| := \sqrt{\langle x, x \rangle}$ . A finite subset  $R$  of  $\mathbb{R}^d \setminus \{0\}$  is called a *root system* if  $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$  and  $\sigma_\alpha(R) = R$  for all  $\alpha \in R$ . For a given root system  $R$ , we denote by  $W$  the finite group generated by all reflections  $\sigma_\alpha$ ,  $\alpha \in R$ . A function  $k : R \rightarrow \mathbb{R}_+$  is called a *multiplicity function* if it satisfies  $k(w\alpha) = k(\alpha)$ , for every  $w \in W$  and every  $\alpha \in R$ .

Throughout this paper we fix a root system  $R$ , a multiplicity function  $k$  and a  $W$ -invariant open subset  $D$  of  $\mathbb{R}^d$ , that is,  $w(D) \subset D$  for all  $w \in W$ . Let  $w_k$  be the *weight function* on  $\mathbb{R}^d$  defined by,

$$w_k(x) := \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{2k(\alpha)},$$

where  $R_+ := \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$  for some  $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in R} H_\alpha$ . Note that  $w_k$  is homogeneous of degree  $2\gamma$ , with  $\gamma := \sum_{\alpha \in R_+} k(\alpha)$ . From now on, we assume that

$$\lambda := \gamma + \frac{d}{2} - 1 > 0.$$

The Dunkl Laplacian associated with the root system  $R$  and the multiplicity function  $k$  is the operator

$$\Delta_k := \sum_{i=1}^d T_i^2,$$

where for every  $1 \leq i \leq d$  and  $f \in C^1(D)$ ,

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad x \in D.$$

By [2], there exists a unique linear isomorphism  $V_k$  from the space of homogenous polynomials of degree  $n$  on  $\mathbb{R}^d$  into it self such that  $V_k 1 = 1$  and  $T_i V_k = V_k \partial_i$ . Later, it was shown in [9] that the intertwining operator  $V_k$  has an homeomorphism extension to  $C^\infty(\mathbb{R}^d)$ . The positivity of  $V_k$  (see [6]) yields the existence of a family of probability measures  $(\mu_x^k)_x$  such that for every  $x \in \mathbb{R}^d$  and every  $f \in C^\infty(\mathbb{R}^d)$ ,

$$V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x^k(y).$$

The support of  $\mu_x^k$  is contained in the convex hull  $C(x)$  of the orbit of  $x$  under the reflection group  $W$ ,

$$C(x) := \text{co}\{wx, \quad w \in W\}.$$

The Dunkl kernel associated with  $R$  and  $k$  is defined on  $\mathbb{R}^d \times \mathbb{R}^d$  by

$$E_k(x, y) := \int_{\mathbb{R}^d} e^{\langle y, \xi \rangle} d\mu_x^k(\xi).$$

It is well known that  $E_k$  is positive, symmetric and admits a unique holomorphic extension to  $\mathbb{C}^d \times \mathbb{C}^d$  satisfying  $E_k(\xi z, \omega) = E_k(z, \xi \omega)$  for every  $z, \omega \in \mathbb{C}^d$  and every  $\xi \in \mathbb{C}$ . The corresponding Dunkl transform is then given for every bounded measure  $\mu$  on  $\mathbb{R}^d$  by

$$\mathcal{F}_D(\mu)(x) := c_k \int_{\mathbb{R}^d} E_k(-i\xi, x) d\mu(\xi), \quad x \in \mathbb{R}^d,$$

where

$$c_k := \left( \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2}} w_k(y) dy \right)^{-1}.$$

If  $\mu = f w_k dx$  where  $f \in S(\mathbb{R}^d)$  and  $dx$  is the Lebesgue measure on  $\mathbb{R}^d$ , then we shall write  $\mathcal{F}_D(f)$  instead of  $\mathcal{F}_D(\mu)$ . Note that  $\mathcal{F}_D$  is injective on the space of all bounded Borel measures  $\mathcal{M}_b(\mathbb{R}^d)$  on  $\mathbb{R}^d$  (see [8]) and is a topological isomorphism from  $S(\mathbb{R}^d)$  into it self (see [3]). For each  $x \in \mathbb{R}^d$ , the Dunkl translation  $\tau_x$  is defined for every  $f \in S(\mathbb{R}^d)$  by

$$\tau_x f = \mathcal{F}_D^{-1}(E_k(ix, \cdot) \mathcal{F}_D f),$$

where  $\mathcal{F}_D^{-1}$  denotes the inverse of  $\mathcal{F}_D$  on  $S(\mathbb{R}^d)$ . In [10], this translation was extended to  $C^\infty(\mathbb{R}^d)$  by

$$\tau_x f(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V_k^{-1} f(z + \eta) d\mu_x^k(z) d\mu_y^k(\eta),$$

where  $V_k^{-1}$  is the inverse of  $V_k$  on  $C^\infty(\mathbb{R}^d)$ . It was shown that, for every  $f \in C^\infty(\mathbb{R}^d)$ , the function  $u : (x, y) \mapsto \tau_x f(y)$  is symmetric, infinitely differentiable on  $\mathbb{R}^d \times \mathbb{R}^d$  and for every  $x, y \in \mathbb{R}^d$ ,

$$(T_i)_x u(x, y) = (T_i)_y u(x, y). \quad (2)$$

Moreover,  $\tau_x f(0) = f(0)$ ,  $T_i \tau_x f = \tau_x T_i f$  and  $\tau_x E_k(z, \cdot)(y) = E_k(x, z) E_k(y, z)$  for every  $z \in \mathbb{C}^d$ . Further, if the support of  $f$  (noted  $\text{supp } f$ ) is in  $B(0, r)$  for some  $r > 0$ , then  $\text{supp } \tau_x f \subset B(0, r + |x|)$ .

According to [7], for each  $x \in \mathbb{R}^d$  and  $r > 0$ , there exists a unique probability measure  $\sigma_{x,r}^k$  on  $\mathbb{R}^d$  which is supported by  $\bigcup_{w \in W} \overline{B}(wx, r) \setminus B(0, ||x| - r|)$  such that for every  $f \in C^\infty(\mathbb{R}^d)$ ,

$$\frac{1}{d_k} \int_{S(0,1)} \tau_x f(ry) w_k(y) d\sigma_{0,1}(y) = \int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y). \quad (3)$$

where,

$$d_k := \int_{S(0,1)} w_k(y) d\sigma_{0,1}(y) = \frac{1}{c_k 2^\lambda \Gamma(\lambda + 1)}.$$

**Lemma 2.1.** *Let  $\varphi \in S(\mathbb{R}^d)$  be radial. Then for every Borel set  $A \subset \mathbb{R}^d$  and every  $x \in \mathbb{R}^d$ ,*

$$\int_A \tau_{-x} \varphi(y) w_k(y) dy = d_k \int_0^\infty \varphi(t) t^{2\lambda+1} \left( \int_A d\sigma_{x,t}^k(y) \right) dt. \quad (4)$$

*Proof.* Let  $x \in \mathbb{R}^d$  and denote by  $\mu(A)$  and  $\nu(A)$  the left hand side and the right hand side respectively of (4). Clearly, both  $\mu$  and  $\nu$  are bounded measures on  $\mathbb{R}^d$ . For every  $y \in \mathbb{R}^d$ ,

$$\begin{aligned} \mathcal{F}_D(\mu)(y) &= \mathcal{F}_D(\tau_{-x}\varphi)(y) \\ &= \mathcal{F}_D\varphi(y) E_k(-iy, x) \\ &= c_k E_k(-iy, x) \int_{\mathbb{R}^d} \varphi(z) E_k(-iy, z) w_k(z) dz \\ &= c_k \int_{\mathbb{R}^d} \tau_x E_k(-iy, \cdot)(z) \varphi(z) w_k(z) dz \end{aligned}$$

Using spherical coordinates and (3), we deduce that,

$$\begin{aligned} \mathcal{F}_D(\mu)(y) &= c_k d_k \int_0^\infty t^{2\lambda+1} \varphi(t) \int_{\mathbb{R}^d} E_k(\xi, -iy) d\sigma_{x,t}^k(\xi) dt \\ &= \mathcal{F}_D(\nu)(y). \end{aligned}$$

Finally, we use the injectivity of  $\mathcal{F}_D$  on  $\mathcal{M}_b(\mathbb{R}^d)$  to conclude.  $\square$

Let  $\varphi \in S(\mathbb{R}^d)$  be a radial function with support in  $\overline{B}(0, r)$ ,  $r > 0$ . We claim that for every  $x \in \mathbb{R}^d$ ,

$$\text{supp } \tau_x \varphi \subset \bigcup_{w \in W} \overline{B}(wx, r). \quad (5)$$

Indeed, let  $A$  be a Borel subset of  $\mathbb{R}^d \setminus \bigcup_{w \in W} \overline{B}(wx, r)$ . Then by (4),

$$\int_A \tau_{-x} \varphi(y) w_k(y) dy = d_k \int_0^r \varphi(t) t^{2\lambda+1} \left( \int_A d\sigma_{x,t}^k(y) \right) dt.$$

Since for every  $0 < t < r$ ,  $\text{supp } \sigma_{x,t}^k \subset \bigcup_{w \in W} \overline{B}(wx, r)$  we deduce that,

$$\int_A \tau_{-x} \varphi(y) w_k(y) dy = 0.$$

This proves the claim.

In the sequel we shall write

$$M_{x,r}(f) = \int_{\mathbb{R}^d} f(y) d\sigma_{x,r}^k(y),$$

whenever the integral makes sense. A Borel function  $f : D \rightarrow \mathbb{R}$  is said to satisfy *the mean value property* on  $D$  if  $M_{x,r}(f) = f(x)$  for every  $x \in \mathbb{R}^d$  and  $r > 0$  such that  $\overline{B}(x, r) \subset D$ .

**Lemma 2.2.** *Let  $f$  be a locally bounded function on  $D$ . If  $f$  satisfies the mean value property on  $D$ , then  $f \in C^\infty(D)$ .*

*Proof.* Without loss of generality we suppose that  $f$  is bounded on  $D$ . Let  $\phi$  be the function defined for every  $t \in \mathbb{R}$  by  $\phi(t) := ce^{-\frac{1}{t}}\chi_{]0,\infty[}(t)$ , where  $\chi_{]0,\infty[}$  is the indicator function of  $]0,\infty[$  and the constant  $c$  is chosen so that

$$cd_k \int_0^1 \phi(1-t^2)t^{2\lambda+1}dt = 1.$$

For every  $n \geq 1$  we define the function  $\phi_n$  by,

$$\phi_n(x) = n^{2\lambda+2}\phi(1-n^2|x|^2), \quad x \in \mathbb{R}^d. \quad (6)$$

Obviously  $\phi_n$  is infinitely differentiable on  $\mathbb{R}^d$  with support in  $\overline{B}(0, \frac{1}{n})$ . Thus, by (5), for every  $x \in \mathbb{R}^d$ ,

$$\text{supp } \tau_x \phi_n \subset \cup_{w \in W} \overline{B}(wx, \frac{1}{n}).$$

Let  $D_n := \{x \in D : \overline{B}(x, \frac{1}{n}) \subset D\}$  and let

$$f_n(x) := \int_D f(y) \tau_{-x} \phi_n(y) w_k(y) dy, \quad x \in \mathbb{R}^d.$$

Then  $f_n \in C^\infty(D_n)$ . On the other hand, it follows from (4) that for every  $x \in D_n$ ,

$$f_n(x) = d_k \int_0^{\frac{1}{n}} \phi_n(t) t^{2\lambda+1} M_{x,t}(f) dt = f(x).$$

Hence  $f \in C^\infty(D_n)$  and consequently  $f \in C^\infty(D)$  as desired.  $\square$

### 3 Main result

$D$  will always denotes a  $W$ -invariant open subset of  $\mathbb{R}^d$ . Our main result is the following:

**Theorem 3.1.** *Let  $f$  be a locally bounded function on  $D$ . The following statements are equivalent:*

- (a)  $f \in C^2(D)$  and  $\Delta_k f = 0$  on  $D$ .
- (b)  $M_{x,r}(f) = f(x)$  for every  $x \in D$  and  $r > 0$  such that  $\overline{B}(x, r) \subset D$ .

The following proposition shows the equivalence between (a) and (b) whenever  $f$  is infinitely differentiable on  $D$ . First, let us recall the Green formula associated with the Dunkl Laplacian (see [4]): For every  $f \in C^2(\overline{B}(0, t))$ ,  $t > 0$ ,

$$\int_{B(0,t)} \Delta_k f(y) w_k(y) dy = \int_{S(0,t)} \frac{\partial}{\partial n} f(y) w_k(y) d\sigma_{0,t}(y), \quad (7)$$

where  $\frac{\partial}{\partial n}$  is the partial derivation operator in the direction of the exterior unit normal.

**Proposition 3.2.** *Assume that  $f$  is infinitely differentiable on  $D$ . Then  $f$  is  $\Delta_k$ -harmonic on  $D$  if and only if  $f$  satisfies the mean value property on  $D$ .*

*Proof.* Let  $x \in D$  and  $r > 0$  such that  $\overline{B}(x, r) \subset D$ . We claim that  $t \mapsto M_{x,t}(f)$  is derivable on  $]0, r[$  and for every  $t \in ]0, r[$ ,

$$\frac{d}{dt}M_{x,t}(f) = \frac{1}{t^{2\lambda+1}} \int_0^t s^{2\lambda+1} M_{x,s}(\Delta_k f) ds. \quad (8)$$

Indeed, since for every  $s \in ]0, r[$ , the support of  $\sigma_{x,s}^k$  is contained in  $\cup_{w \in W} B(wx, r)$ , it suffices to prove (8) replacing  $f$  by a function  $h \in C^\infty(\mathbb{R}^d)$  such that

$$h = f \text{ on } \cup_{w \in W} B(wx, r).$$

It is easily seen from (3) that for every  $t \in ]0, r[$ ,

$$\begin{aligned} \frac{d}{dt}M_{x,t}(h) &= \frac{1}{d_k} \int_{S(0,1)} \langle \nabla(\tau_x h)(ty), y \rangle w_k(y) d\sigma_{0,1}(y) \\ &= \frac{1}{d_k t^{2\lambda+1}} \int_{S(0,t)} \langle \nabla(\tau_x h)(u), \frac{u}{t} \rangle w_k(u) d\sigma_{0,t}(u) \\ &= \frac{1}{d_k t^{2\lambda+1}} \int_{S(0,t)} \frac{\partial}{\partial n}(\tau_x h)(u) w_k(u) d\sigma_{0,t}(u). \end{aligned}$$

Therefore, by the Green formula (7) and the fact that  $\Delta_k \tau_x = \tau_x \Delta_k$ ,

$$\frac{d}{dt}M_{x,t}(h) = \frac{1}{d_k t^{2\lambda+1}} \int_{B(0,t)} \tau_x(\Delta_k h)(u) w_k(u) du.$$

Hence, using spherical coordinates we deduce that,

$$\frac{d}{dt}M_{x,t}(h) = \frac{1}{t^{2\lambda+1}} \int_0^t s^{2\lambda+1} M_{x,s}(\Delta_k h) ds.$$

Thus the claim is proved. Now assume that  $\Delta_k f = 0$  on  $D$ . Then for all  $t \in ]0, r[$ ,  $\frac{d}{dt}M_{x,t}(f) = 0$ , by (8). This yields that  $M_{x,t}(f) = \lim_{s \rightarrow 0} M_{x,s}(f)$ . On the other hand, it is known from [7] that the map  $(x, s) \mapsto \sigma_{x,s}^k$  is continuous with respect to the weak topology on  $\mathcal{M}_b(\mathbb{R}^d)$ . Thus,

$$\lim_{s \rightarrow 0} M_{x,s}(f) = f(x). \quad (9)$$

Whence  $M_{x,t}(f) = f(x)$  which yields the necessity. Conversely, assume that  $f$  satisfies the mean value property on  $D$ . Then, using (8) we deduce that  $M_{x,t}(\Delta_k f) = 0$  for all  $t \in ]0, r[$ . Letting  $t$  tend to 0 we obtain that  $\Delta_k f(x) = 0$ .  $\square$

We then conclude, in virtue of Lemma 2.2, that every locally bounded function  $f$  on  $D$  which satisfies the mean value property on  $D$  is necessarily  $\Delta_k$ -harmonic on  $D$ . The converse statement will be proved in the remainder of this section.

**Lemma 3.3.** *Let  $(h_n)_{n \geq 1} \subset C^\infty(D)$  be a locally uniformly bounded sequence of  $\Delta_k$ -harmonic functions on  $D$  with pointwise limit  $h$ . Then  $h \in C^\infty(D)$  and  $\Delta_k h = 0$  on  $D$ .*

*Proof.* Let  $x \in D$  and let  $r > 0$  such that  $\overline{B}(x, r) \subset D$ . Since for every  $n \geq 1$  the function  $h_n$  is  $\Delta_k$ -harmonic on  $D$ , it follows from Proposition 3.2 that,

$$h_n(x) = \int_{\mathbb{R}^d} h_n(y) d\sigma_{x,r}^k(y).$$

Applying the dominated convergence theorem, we get  $h(x) = M_{x,r}(h)$ . Whence  $h$  satisfies the mean value property on  $D$  which finishes the proof, by Lemma 2.2 and Proposition 3.2.  $\square$

Let  $g_k$  be the fundamental solution of the Dunkl Laplacian. That is, for every  $\varphi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\int_{\mathbb{R}^d} g_k(y) \Delta_k \varphi(y) w_k(y) dy = -\varphi(0). \quad (10)$$

It is well known from [4] that,

$$g_k(y) = c_k \Gamma(\lambda) 2^{\lambda-1} |y|^{-2\lambda}. \quad (11)$$

**Theorem 3.4.** *Let  $h \in C(D)$  and  $f \in C^\infty(D)$ . Assume that for every  $\varphi \in C_c^\infty(D)$ ,*

$$\int_D h(x) \Delta_k \varphi(x) w_k(x) dx = \int_D f(x) \varphi(x) w_k(x) dx.$$

*Then  $h \in C^\infty(D)$ .*

*Proof.* It suffices to prove that  $h \in C^\infty(U)$ , for every  $W$ -invariant open set  $U$  such that  $\overline{U} \subset D$ .

*Step 1.* Assume first that  $f = 0$  on  $D$ . Choose  $n_0 \geq 1$  such that for every  $x \in U$ ,  $\overline{B}(x, \frac{1}{n_0}) \subset D$ . For every  $n \geq n_0$ , let  $\phi_n$  be as in (6). Then, the function  $h_n$  defined on  $U$  by

$$h_n(x) := \int_D h(y) \tau_{-x} \phi_n(y) w_k(y) dy,$$

is infinitely differentiable on  $U$  and by (2) for every  $x \in U$ ,

$$\Delta_k h_n = \int_D h(y) \Delta_k (\tau_{-x} \phi_n)(y) w_k(y) dy.$$

On the other hand, it follows from (4) that for every  $x \in U$ ,

$$\begin{aligned} h_n(x) &= d_k \int_0^{\frac{1}{n}} \phi_n(t) t^{2\lambda+1} M_{x,t}(h) dt \\ &= c d_k \int_0^1 \phi(1-u^2) u^{2\lambda+1} M_{x, \frac{u}{n}}(h) du. \end{aligned}$$



This yields that  $(h_n)_{n \geq n_0}$  is uniformly bounded on  $U$  and converges pointwise to  $h$  on  $U$ . Hence, in view of Lemma 3.3,  $h \in C^\infty(U)$  and  $\Delta_k h = 0$  on  $U$ .

*Step 2.* We now turn to the general case where  $f$  is not trivial. Let  $v \in C_c^\infty(\mathbb{R}^d)$  such that  $v = f$  on  $U$  and define  $\psi$  on  $\mathbb{R}^d$  by

$$\psi(x) := \int_{\mathbb{R}^d} g_k(y) \tau_x v(y) w_k(y) dy,$$

where  $g_k$  is given by (11). Using spherical coordinates, it easily seen that the function  $g_k w_k$  is locally Lebesgue integrable on  $\mathbb{R}^d$ . Thus,  $\psi \in C^\infty(\mathbb{R}^d)$ . Furthermore, it follows from (2) and (10) that  $\Delta_k \psi = -f$  on  $U$ . Then, for every  $\varphi \in C_c^\infty(U)$ ,

$$\int_{\mathbb{R}^d} (h(x) + \psi(x)) \Delta_k \varphi(x) w_k(x) dx = \int_{\mathbb{R}^d} (f(x) + \Delta_k \psi(x)) \varphi(x) w_k(x) dx = 0.$$

Whence, the first step yields that  $h + \psi$  is infinitely differentiable on  $U$  which finishes the proof.  $\square$

We note that the previous theorem was already proved by H. Mejjajalli and K. Trimèche [5] using Sobolev spaces associated with the Dunkl operators.

*Proof of Theorem 3.1* Statement (a) follows from (b) by means of Lemma 2.2 and Proposition 3.2. Assume now that (a) holds. Then, by [?], for every  $\varphi \in C_c^\infty(D)$ ,

$$\int_D f(x) \Delta_k \varphi(x) w_k(x) dx = \int_D \Delta_k f(x) \varphi(x) w_k(x) dx = 0.$$

Use now Theorem 3.4 and Proposition 3.2 to finish the proof.  $\square$

In the following we shall give a counterexample proving that Theorem 3.1 does not hold true if the open set  $D$  is not  $W$ -invariant. To that end, let  $d = 1$  and consider the root system  $R = \{\pm\sqrt{2}\}$ . Then, the corresponding reflection group is given by  $W = \{\pm id_{\mathbb{R}}\}$ . Therefore, an open set  $U \subset \mathbb{R}$  is  $W$ -invariant if and only if it is symmetric.

**Proposition 3.5.** *For every non symmetric open set  $U \subset \mathbb{R}$  there exists a function  $h : \mathbb{R} \rightarrow \mathbb{R}$  which is  $\Delta_k$ -harmonic on  $U$  but does not satisfy the mean value property on  $U$ .*

*Proof.* To abbreviate the notation we write  $I_{x,r} := ]x - r, x + r[$ . Let  $x \in U$  and  $r > 0$  such that  $\bar{I}_{x,r} \subset U$  and  $\bar{I}_{-x,r} \cap \bar{U} = \emptyset$ . Choose  $f \in C_c^\infty(\mathbb{R}^d)$  such that  $f = -1$  on  $\bar{I}_{-x,r}$  and  $f = 0$  on  $U$ . Since  $g_k w_k$  is locally Lebesgue integrable, we deduce that the function  $h$  defined on  $\mathbb{R}$  by

$$h(z) = \int_{\mathbb{R}} g_k(y) \tau_z f(y) w_k(y) dy,$$

is infinitely differentiable on  $\mathbb{R}$ . Moreover, by (10), for every  $z \in \mathbb{R}$ ,

$$\Delta_k h(z) = -\tau_z f(0) = -f(z).$$

Hence,  $\Delta_k h = 0$  on  $U$ . On the other hand, for every  $t \in ]0, r[$ ,

$$M_{x,t}(\Delta_k h) = - \int_{\mathbb{R}} f(y) d\sigma_{x,t}^k(y) = \sigma_{x,t}^k(\bar{I}_{-x,t}).$$

Moreover, it follows from [7, Remarks 4.2] that

$$\text{supp } \sigma_{x,t}^k = \bar{I}_{x,t} \cup \bar{I}_{-x,t}.$$

Thus  $\sigma_{x,t}^k(\bar{I}_{-x,t}) > 0$  and consequently  $M_{x,t}(\Delta_k h) > 0$ . Whence, by (8) the function  $t \mapsto \frac{d}{dt} M_{x,t}(h)$  is positive on  $]0, r[$ . Hence,  $M_{x,t}(h) \neq h(x)$  for every  $t \in ]0, r[$ , which means that  $h$  does not satisfy the mean value property on  $U$ .  $\square$

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